The only explanation which can at present be advanced to account for the difference between the observations of Deville and Troost on the one hand, and Meyer and Crafts on the other, is that in the experiments of the former the iodine was gradually converted into vapour, whereas the method adopted by the latter involves the almost instantaneous volatilisation of the iodine; in the case of some organic compounds a difference of this kind in the mode of heating is known to exercise a considerable and in many respects similar influence on the result, so that this explanation is not unsupported by analogy.

Great difficulty was experienced in determining the density of free bromine in consequence of the explosive rapidity with which it is converted into gas when introduced into the intensely-heated bulb of the density apparatus. The results obtained are not accordant, but all lie between the number corresponding to the molecular formula $Br_2$ and that required on the assumption that dissociation takes place to the same extent as in the case of iodine. Using platinic bromide, $PtBr_4$, however, instead of free bromine, Meyer and Züblin find that a reduction in density takes place precisely of the character of that observed for chlorine from platinous chloride and for iodine. Thus at a temperature of about 1,570° the observed density in two experiments was 3·78 and 3·64, 3·64 being exactly two-thirds of the density corresponding to the molecular formula $Br_2$.

As yet Meyer has told us nothing of the nature of the dissociation products of the three halogens; their determination and separation will probably be attended with great experimental difficulties, but the problem could not well be placed in abler hands, and we trust that ere long we may be able to congratulate him on the accomplishment of this the crowning triumph of his labours.

HENRY E. ARMSTRONG

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## GLAISHER'S FACTOR TABLES

*Factor Table for the Fourth Million.* By James Glaisher, F.R.S. (London: Taylor and Francis, 1880.)

THERE is no general method of ascertaining whether one number is divisible, without remainder, by another specified number (less than its half) except by actual trial, or by the knowledge, otherwise acquired, of all the divisors of the first number. If then the second is not among these, it is also known that it is not a divisor of the first number. The knowledge of whether a specified number has any divisors at all, and if so what they are, is only to be obtained in general by trying it with all possible divisors less than its square root. The process can be shortened, but only to a limited extent, and, speaking generally, it would require hundreds of division sums, to ascertain by trial that 3,979,769 had 1979 for a divisor, and was consequently the product of 1979 and 2011.

It is, however, frequently important to mathematicians to know how to split up any given number into its divisors or factors, and this without the enormous labour which *may* be involved in actually trying for its divisors, especially as there is no general mathematical principle which enables us to dispense with the trial, or even to shorten it so as to bring it within practicable limits. The alternative is to tabulate numbers up to a given limit, and to indicate, for each, whether it has divisors, and what they are. It is not necessary, or usual, to include in such tables *every* number without exception; for an inspection of the last figure of any number tells us whether it is divisible by *two* or by *five;* and the old rule of "casting out the nines" tells us whether it is divisible by *three*. These considerations greatly reduce the number which it is necessary to tabulate; for, among the first 300 numbers, 150 are even, that is to say, divisible by 2; and of the remaining 150, 50 are divisible by 3; while of the 100 left after that, 20 are divisible by 5. The exclusion of the numbers divisible by 2, 3, or 5 thus reduces the number of tabular entries required, from 300 to 80, and this proportion holds all through the table. as well as for the first 300 numbers. It will be observed that the last two figures of these 80 numbers remain the same for every batch of 300. This facilitates the tabulation, and advantage has been taken of this facility in printing the Tables.

The first extensive tables of this kind were those published by the Austrian General, Baron von Vega, at the close of the last century. These extended from 1 to 108,000, and thus give all the divisors of the numbers not divisible by 2, 3, and 5 within those limits. The next table was that of Chernac, a Polish Professor of Mathematics at Deventer, in Over-yssel, which was published in 1811. It contained all the divisors of all numbers, not divisible by 2, 3, and 5 up to 1,012,000. It forms a very thick quarto volume of over 1,000 pages.

The next extension was made by Burckhardt (1814-17), who published a series of three volumes, giving, not *all* the divisors, but the least prime divisor, of all numbers (except those divisible by 2, 3, and 5) up to 3,036,000. This is not quite so convenient, as a matter of immediate reference, as giving all the divisors; but it answers every necessary purpose. For example, when we know that 3,999,589 has 11 for its least divisor, we can find by actual division that the quotient is 363,599. We "look out" this number in the earlier part of the table, being sure of finding it there, seeing that 11 was the *least* divisor of its multiple; we find its least divisor to be 31. Performing the division by 31, we obtain the quotient 11,729. We "look out" this again in the earlier part of the table, and we find that 37 is the least divisor. Performing this division, we obtain 317 as the quotient. Since this is less than 37 × 37, we know that it can have no divisors except unity and itself, or that it is prime. If, instead of the least prime divisor, all the divisors had been given, we should at once have found from the table

$$3{,}999{,}589 = 11 \times 31 \times 37 \times 317.$$

There is an obvious advantage in the more complete table. Unfortunately it is balanced by the practical inconvenience of size, and "a great book is a great evil." What this practically comes to may be judged of from the remark that Chernac's table, which gives all the prime factors from 1 to 1,019,000, takes 1020 quarto pages; while Burckhardt's, which gives only the least prime divisor, contains the numbers from 1 to 3,036,000 in 336 quarto pages. It is true that Burckhardt's table is more closely printed than Chernac's, with somewhat smaller type, and a slightly larger form; but, making all allowances, the condensation obtained by giving the least prime divisor instead of all the divisors, cannot be put at less than three to one. It must be observed also, that the processes of division, which have to be performed when the least divisors only are given, are definite divisions with a known divisor, and do not involve the tentative process of finding what the divisor is; it is just this tentative process which it is the object of all such tables to avoid.

Burckhardt's tables extend, as has already been stated, to 3,036,000. They consequently tabulate no divisor exceeding 1741, which is the prime number next below the square root of 3,036,000, which lies between 1742 and 1743.

The celebrated German computer, Zacharias Dase, began the task of extending this table to nine or ten millions. There was then a prospect of the fourth, fifth, and sixth millions being printed from a manuscript by Crelle, so that Dase, instead of taking the fourth million, began with the seventh. His task was interrupted by his death, but was resumed by his friend Dr. Rosenberg. The seventh, eighth, and ninth millions were published by a Society called the Dase-Verein, and printed in Hamburg, and the tenth million exists in manuscript. These publications, however, were of little practical value to science, so long as the gap between 3,036,000 (Burckhardt's final limit), and 7,000,000 (Dase's initial limit) remained a blank, and it was found that Crelle's manuscript, which had fallen into the possession of the Berlin Academy, was not sufficiently reliable, in respect of accuracy, to supply this gap.

This blank Mr. Glaisher has undertaken to fill up, and the first instalment, the fourth million, is now before us. With the assistance, towards the expenses of computation and printing, of grants from the British Association, and of the Government grants administered by the Royal Society, but without any requital of his own toil, except such as all good workers find in the satisfactory completion of their labour, he has secured for England a share in the performance of this work. The fourth million, added to Burckhardt's three millions, makes perfect work as far as it goes. The fifth million is now going through the press, and the manuscript of the sixth million is nearly complete. When these are printed, the work of Dase and Rosenberg will couple on, and we shall have, in a shape available for immediate reference, a complete knowledge of the divisors of all numbers up to nine millions. To test a number nearly equal to nine millions might involve our trying, as divisors, all the prime numbers from 7 to 2,999 inclusive.

It would be premature to discuss the question of accuracy of performance until much more trial has been made of the work than has been possible in the few days which have elapsed since its appearance. Very good guarantees, however, are afforded by the systematic method in which the process of calculation has been performed, as well as by the great experience which Mr. Glaisher has had in accurate computation, and again by the numerical tests of comparing the number of primes actually counted, within given limits, with the approximate numbers indicated by theory.

It is well known that the frequency of the occurrence of prime numbers in the neighbourhood of any large number, $x$, is expressed by the reciprocal of the hyperbolic logarithm of $x$. Soldners' integral, $\int_0^x \frac{dx}{\log x}$, should therefore express, with a high degree of approximation when $x$ is large, the number of primes below a certain number. One difficulty of the application of this is, that the function integrated becomes infinite between the limits. Nevertheless a highly approximate formula for the number of primes below a high number $x$ is given by the expression (due to Legendre)—

$$\frac{x}{\log x - 1{\cdot}08366}.$$

A serious practical difficulty in attributing exactness to any such formula, or in determining its constants to any high approximation, lies in the irregular distribution of the prime numbers. It not unfrequently happens that two consecutive odd numbers are primes; that is so with 3,999,311 and 3,999,313. On the other hand there is no prime number at all between 3,826,019 and 3,826,157, which differ by 138. This variation of frequency effectually throws out any minute comparison between actual counting, and analytical expressions for the number of primes, founded on the assumption of regular continuity. A discussion of this part of the subject is given in Mr. Glaisher's introduction.

For a full development of this and of the cognate theorems, and of their limits, we must refer to the extremely valuable preface which Mr. Glaisher has prefixed to his work. To that also we must refer for an account of the ingenious methods used in abridging the enormous labour of computation, and at the same time of seizing the advantages of the most systematic arrangement possible, in order to secure accurate work in the first instance, and then the detection of error, if accidentally committed. The amount of accuracy which it is possible to obtain may be inferred from the fact, that after many years' use, only two errors have yet been pointed out in Burckhardt's extensive table, and that Chernac is nearly as good. We have no reason to doubt that this high standard of accuracy has been maintained by Mr. Glaisher.

Our review would hardly be complete without some remarks on the utility of this work. We have already pointed out the utter impossibility, as a practical question to practical men, of ascertaining whether a given number has divisors, and what they are, without the help of such tables. One of the most obvious applications is to the calculation of high logarithms. The larger logarithmic tables, to a great number of figures, only extend from 1 to 20,000 in the case of common logarithms, and from 1 to 10,000 in the case of Napierian logarithms. When, therefore, such a logarithm is required for an incommensurable number (as is commonly the case), it becomes necessary to split it, either absolutely or approximately, into factors. Now this series of tables, when complete, will give us at sight the breaking up of the first seven figures of any number, and by a little adjustment, turning upon the formula $a^2 - (b \pm c)^2$, suggested by Burckhardt, to a far higher extent. For instance, Burckhardt himself gives as an approximate value of $\pi$, $256 . 19 . 173 . 229 . 509 . 3203$, which (neglecting cyphers) is good for the first ten figures; and in the same way it has been found that

$$\log_e 10 = 64 . 5 . 13 . 13 . 103 . 109 . 541 . 701 + 24844$$

up to the fourteenth decimal place.

Again, the tables virtually furnish lists of primes to their full extent. We need not remind mathematical readers how often it is important to know, concerning certain results of calculation, whether a number is prime or not, this being the necessary preliminary to further inferences from the processes which give rise to it. As an easy example of the consequence of knowing how a number splits up into prime factors, we may mention the elementary theorem, that any recurring decimal whose period consists of five figures *must* have one or more of the numbers 3, 41, or 271, as factors of its divisor. This is simply a consequence of the numerical identity—

$$99999 = 3 . 3 . 41 . 271.$$

Now, as to relative utility: We are inclined to think that the utility of such tables is measured by the index of the power of 10, to which they extend—that this rule represents the advantage of Vega's table, up to 108,000, over Barlow's, up to 10,000; of Chernac's, up to 1,012,000, over Vega's; and of this set of tables, when completed up to ten millions, over Chernac's. We think this estimate holds for theoretical questions relating to the enumeration and distribution of primes, and cognate questions relating to the theory of numbers, as well as for the practical command they give us over the numbers themselves. Nevertheless, it would not be right to underestimate the important point that this work *does* give us a command over numerical magnitude such as we did not possess before. In that view he would be a bold man who should say that the money cost of the production and printing of these tables was a bad investment for science, especially when the directing labour was gratuitously given. What that directing labour involves can be understood by those alone who have worked upon millions. None others know what an awful factor a million is, when applied to the multiplication of the simplest process. We shall heartily congratulate Mr. Glaisher on the termination of his labours, and we no less heartily congratulate our mathematical friends upon their good fortune in having found such a man to undertake such a task.

We conclude by reminding our mathematical readers that all the processes by which these tables have been formed are but skilful adaptations of the well-known CRIBRUM ERATOSTHENIS, of which the analytical expression was first given by Euler (*introductio in analysin infinitorum*) in his remark that the harmonic series—

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \ldots \text{ to infinity}$$

is the reciprocal of the continued product—

$$(1 - \tfrac{1}{2})(1 - \tfrac{1}{3})(1 - \tfrac{1}{5})(1 - \tfrac{1}{7}) \ldots,$$

in which the primes only enter.

C. W. M.

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## *WHO ARE THE IRISH?*

*Who are the Irish?* By James Bonwick, F.R.G.S. (London: David Bogue, 1880.)

THIS little work is issued as the first of a series on "Our Nationalities," to be followed by three others on the Scotch, Welsh, and English. It does not appear from the prospectus whether the rest of the series is to be entrusted to Mr. Bonwick; but if they are it is to be hoped that he will qualify himself for the task by a preliminary study of at least the first principles of ethnology. The present volume, with all its good intentions and praiseworthy industry, must be regarded as a hopeless failure, owing entirely to the neglect of this necessary precaution. For many years ethnology, anthropology, and philology were subjects which any one seemed competent to deal with, who had got hold of a few lists of words in some obscure African or Polynesian dialects (the obscurer the better), or who had desecrated a sufficient number of ancient barrows, or posed to admiring circles under the shadow of some Druid's altar in Cornwall or Brittany. But those halcyon days of the amateur ethnologist are no more, though the writer, unfortunately, seems scarcely alive to the fact. Almost every page of his little tractate betrays solecisms and crudities, such as one naturally looks for in the writings of the Pinkertons, Vallanceys, Vans Kennedys, Bethams, and other obsolete writers of the old Keltic school, but which have become anachronisms since Keltic studies have been placed on a solid basis by the labours of Pritchard, Pictet, Zeuss, Ebel, Lottner, Diefenbach, Whitley Stokes, and Dr. W. K. Sullivan.

A great many authorities are quoted, some, it may be, at first hand, but most of them vicariously, some good, some of no account, some utterly worthless. But all are treated with equal deference, and nowhere is there betrayed the least sense of discrimination as to their respective merits. Thus at p. 27 we have "Betham makes them Teutons, and Wilde, Celts," as if the opinion of a keltomaniac like Sir William Betham could matter a straw one way or the other, and as if in the writer's view it commanded as much weight as that of the distinguished member of the Royal Irish Academy, with whom he is here strangely associated. This vice pervades the entire work, and of itself alone reveals the utter incapacity of the author to deal with such a theme as that of the affinities of the Irish race. Hence it is not perhaps surprising to find ethnical terms treated quite as wildly as ethnological authorities. At p. 19 occurs the following passage, which is quite a curiosity in its way:—"The Basques are believed to be of Turanian origin, while the Celts are Aryans, like most of the Europeans, as well as Persians; Hindoos, &c. Some Turkish and Finnish tribes, with ancient races in Greece, Italy, and Assyria, have been deemed Turanian with Tartar (*sic*) sympathies. The Etruscans of Tuscany were leaning to the Iberian." For wild incoherence and confusion this will surely hold its own with anything to be found in the lucubrations of the most popular exponents of Keltic ethnology in the present or past generation. Frequent use is naturally made of the convenient but dangerous term "Turanian," but its meaning is nowhere defined. Careful writers, if they use it at all, at least restrict it to the Finno-Tataric or Ural-Altaic family. But it is here apparently separated from that connection, so far at least as regards the Tatars, while the Tatars themselves are spoken of as something distinct from the "Turkish" (read Tûrki) tribes, with whom they are nevertheless identical. Why or when "the Etruscans of Tuscany were leaning to the Iberian" we are not informed, nor are we told by whom "the Basques are believed to be of Turanian origin." Meantime it may be well to remind the author